Conformal Field Theory and Gravity

Solutions to Problem Set 6 Fall 2024

1. A stack of strings

(a)

$$S_{\text{Poly}}[G] = -\frac{1}{4\pi\alpha'}\int d^2\sigma\; G_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu g^{\alpha\beta} \tag{1}$$

$$S_{NG}[G] = -\frac{1}{2\pi\alpha'}\int d^2\sigma\; \sqrt{-\det(\gamma_{\alpha\beta})} \tag{2}$$

where we defined the induced metric $\gamma_{\alpha\beta} \equiv G_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu$.

(b) In static gauge, we find the induced metric

$$\gamma_{tt} = -f(r)^{-1} + \frac{d\vec{X}}{dt}\cdot\frac{d\vec{X}}{dt} \qquad \gamma_{xx} = f(r)^{-1} + \frac{d\vec{X}}{dx}\cdot\frac{d\vec{X}}{dx} \qquad \gamma_{xt} = \frac{d\vec{X}}{dt}\cdot\frac{d\vec{X}}{dx} \tag{3}$$

Hence we find

$$-\det(\gamma_{\alpha\beta}) = -\gamma_{tt}\gamma_{xx} + \gamma_{xt}^2 = f(r)^{-2} - f(r)^{-1}\Big(\frac{d\vec{X}}{dt}\cdot\frac{d\vec{X}}{dt} - \frac{d\vec{X}}{dx}\cdot\frac{d\vec{X}}{dx}\Big) + \mathcal{O}(X^4) \tag{4}$$

By plugging this into the NG action and expanding up to leading order in powers of $\vec{X}$, we get the result

$$L \approx \frac{1}{2\pi\alpha'}\int dt dx\Big[-f(r)^{-1} + \frac{1}{2}\Big(\frac{d\vec{X}}{dt}\cdot\frac{d\vec{X}}{dt} - \frac{d\vec{X}}{dx}\cdot\frac{d\vec{X}}{dx}\Big) + ...\Big] \tag{5}$$

The corresponding Euler-Lagrange equations are

$$\frac{d^2\vec{X}}{dt^2} - \frac{d^2\vec{X}}{dx^2} = -\vec{\nabla}(f(r)^{-1}) = -22 g_s^2 N \frac{\left(\frac{l_s}{r}\right)^{22}}{f(r)^2}\frac{\vec{X}}{r} \tag{6}$$

Therefore, $f(r)^{-1}$ generated by the stack of strings acts as an attractive potential on the probe string.

(c) The $B-$field action yields

$$\frac{1}{4\pi\alpha'}\int d^2\sigma\; 2(f(r)^{-1} - 1) \tag{7}$$

Hence the total action is

$$S = \frac{1}{2\pi\alpha'}\int dt dx\Big[-1 + \frac{1}{2}\Big(\frac{d\vec{X}}{dt}\cdot\frac{d\vec{X}}{dt} - \frac{d\vec{X}}{dx}\cdot\frac{d\vec{X}}{dx}\Big) + ...\Big] \tag{8}$$

Hence the string now feels no total force.

2. The different regimes of the Dp-brane gravity description

(a) When $p > 3$ and $\alpha' \to 0$, keeping $g_{\rm YM}$ fixed means that $g_s \to \infty$. In this regime, we would need a non-perturbative definition of string theory. The only method to address such case is to use S-duality, which is a strong-weak duality that relates string theory with coupling g_s to string theory with coupling $1/g_s$.

(b) First, using spherical coordinates for the transverse directions with radius $r = U\alpha'$, the metric can be written as

$$ds^2 = f_p^{-1/2}(-dt^2 + dx_1^2 + ...dx_p^2) + \alpha'^2 f_p^{1/2} dU^2 + \alpha'^2 f_p^{1/2} U^2 d\Omega_{8-p} \tag{9}$$

Then, using that in the limit $\alpha' \to 0$

$$f_p \to \frac{1}{\alpha'^2} \frac{d_p N g_{\rm YM}^2}{U^{7-p}} \tag{10}$$

we find the given result,

$$ds^2 = \alpha' \left(\frac{U^{(7-p)/2}}{g_{\rm YM}\sqrt{d_p N}} dx_{||}^2 + \frac{g_{\rm YM}\sqrt{d_p N}}{U^{(7-p)/2}} dU^2 + g_{\rm YM}\sqrt{d_p N} U^{(p-3)/2} d\Omega_{8-p}^2 \right). \tag{11}$$

Regarding the dilaton, we usually define g_s as

$$g_s = e^{\phi_\infty} \tag{12}$$

Thus,

$$e^\phi = g_s f_p^{(3-p)/4} \tag{13}$$

Taking the limit $\alpha' \to 0$,

$$e^\phi \to g_s \alpha'^{(p-3)/2} \left(\frac{d_p N g_{\rm YM}^2}{U^{7-p}} \right)^{(3-p)/4} = g_{\rm YM}^2 (2\pi)^{2-p} \left(\frac{d_p N g_{\rm YM}^2}{U^{7-p}} \right)^{(3-p)/4} \tag{14}$$

(c) Let us first consider a generic case where we have a metric of the form

$$ds^2 = \underbrace{g_{mn}(x^p) dx^m dx^n}_{\mathcal{M}_0} + f(U) \underbrace{g_{ab}(x^c) dx^a dx^b}_{\mathcal{M}_1} \tag{15}$$

where U is one of the coordinates x^m and we separated the indices in the two manifolds, $m, n, ...$ for $\mathcal{M}_0$ and $a, b, ...$ for $\mathcal{M}_1$. Let us also assume that

$$g_{mn} dx^m dx^n = g_{UU}(U) dU^2 + g_{m'n'} dx^{m'} dx^{n'} \tag{16}$$

where $x^{m'}$ are all the coordinates x^m different from U. We will compute the Ricci scalar using

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\kappa\lambda} (\partial_\mu g_{\kappa\nu} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu}) \tag{17}$$

$$R = g^{\nu\sigma} (\partial_\mu \Gamma^\mu_{\sigma\nu} - \partial_\sigma \Gamma^\mu_{\mu\nu} + \Gamma^\lambda_{\sigma\nu} \Gamma^\mu_{\mu\lambda} - \Gamma^\lambda_{\mu\nu} \Gamma^\mu_{\sigma\lambda}) \tag{18}$$

To do so, we will separate R into three contributions. The Ricci scalar R_0 associated to g_{mn}, the Ricci scalar R_1 associated to g_{ab}, and the remaining contribution from the non-trivial relation between U and $\mathcal{M}_2$. The non-trivial interpolating Christoffel symbols between $\mathcal{M}_1$ and $\mathcal{M}_2$ are

$$\Gamma^U_{ab} = -\frac{1}{2} g^{UU} g_{ab} f'(U) \qquad \Gamma^b_{Ua} = \frac{1}{2}\delta^b_a \frac{f'(U)}{f(U)} = \Gamma^b_{aU} \tag{19}$$

These are the only contributions in R that are due to the cross-terms between $\mathcal{M}_0$ and $\mathcal{M}_1$. We thus obtain

$$R = R_0 + \frac{1}{f} R_1 + \text{crossterms} \tag{20}$$

where R_0 is the Ricci scalar of g_{mn} and R_1 is the Ricci scalar of g_{ab}. One finds that the crossterms are,

$$\begin{aligned}
\text{crossterms}(d_1, f) =& \frac{1}{f} g^{ab}\partial_U \Gamma^U_{ab} - g^{UU}\partial_U \Gamma^a_{aU} + \frac{1}{f} g^{cd}\Gamma^U_{cd}\Gamma^a_{aU} - g^{UU}\Gamma^b_{aU}\Gamma^a_{Ub} \\
& - \frac{1}{f} g^{bc}\Gamma^d_{Ub}\Gamma^U_{cd} - \frac{1}{f} g^{bc}\Gamma^U_{ab}\Gamma^a_{cU} \\
=& -\frac{d_1}{2f}(g^{UU} f')' - \frac{d_1}{2} g^{UU}\left(\frac{f'}{f}\right)' - \frac{d_1^2}{4} g^{UU}\left(\frac{f'}{f}\right)^2 + \frac{d_1}{4} g^{UU}\left(\frac{f'}{f}\right)^2
\end{aligned} \tag{21}$$

where d_1 is the dimension of $\mathcal{M}_1$.

This argument can of course be applied recursively to a metric of the type

$$ds^2 = g_{mn} dx^m dx^n + f_1(U) g_{a_1 b_1} dx^{a_1} dx^{b_1} + f_2(U) g_{a_2 b_2} dx^{a_2} dx^{b_2} + \ldots \tag{22}$$

In which case

$$R = R_0 + \frac{1}{f_1} R_1 + \frac{1}{f_2} R_2 + \cdots + \text{crossterms}(d_1, f_1) + \text{crossterms}(d_2, f_2) + \ldots \tag{23}$$

where $d_1, d_2, \ldots$ are the dimensions of $\mathcal{M}_1, \mathcal{M}_2, \ldots$

Our metric has exactly this form, namely

$$ds^2 = \underbrace{\alpha' \frac{g_{\rm YM}\sqrt{d_p N}}{U^{(7-p)/2}}}_{g_{UU}} dU^2 + \underbrace{\alpha' g_{\rm YM}\sqrt{d_p N}\, U^{(p-3)/2}}_{f_1(U)} d\Omega^2_{8-p} + \underbrace{\alpha' \frac{U^{(7-p)/2}}{g_{\rm YM}\sqrt{d_p N}}}_{f_2(U)} dx^2_{||}\,. \tag{24}$$

The Ricci scalar of dU and of $dx_{||}$ are vanishing, $R_0 = R_2 = 0$, whereas the Ricci scalar of the $(8-p)$-sphere is $R_1 = (8-p)(7-p)$. We thus get

$$\begin{aligned}
R =& \frac{1}{f_1} R_1 + \text{crossterms}(d_1 = 8-p, f_1) + \text{crossterms}(d_2 = p+1, f_2) \\
=& \frac{(3-p)(6-p)(7-p)(p+1)}{8\alpha' g_{\rm YM}\sqrt{d_p N}} U^{(3-p)/2} \sim \frac{1}{\alpha'}\frac{1}{g_{\rm eff}}
\end{aligned} \tag{25}$$

(d) The condition $g^2_{\rm eff} \gg 1$ follows automatically from $\alpha' R \ll 1$, whereas the condition on the dilaton gives

$$e^\phi \sim g^2_{\rm YM}\left(\frac{g^2_{\rm YM} N}{U^{7-p}}\right)^{(3-p)/4} \sim \frac{g^2_{\rm eff}}{N}(g^2_{\rm eff})^{(3-p)/4} \ll 1 \implies g^2_{\rm eff} \ll N^{4/(7-p)} \tag{26}$$

(e) Let us start with the famous CFT case $p = 3$.

$$\alpha' R \ll 1 \implies g_{\rm YM}\sqrt{N} \equiv \sqrt{\lambda} \gg 1 \tag{27}$$

$$e^{\phi} \sim g_{\rm YM}^2 = \frac{1}{N}\lambda \ll 1 \tag{28}$$

where $\lambda = N g_{\rm YM}^2$ is the 't Hooft coupling. Combining both conditions implies

$$\lambda \gg 1 \qquad N \gg 1 \tag{29}$$

For $p \neq 3$, the condition of small curvatures gives

$$\alpha' R \sim \frac{U^{(3-p)/2}}{g_{\rm YM}\sqrt{N}} \ll 1 \tag{30}$$

When $p < 3$, we see that the curvature grows with U, i.e. U cannot be too big. When $p > 3$, this means that the curvature decreases with U, i.e. U cannot be too small.

$$\text{Small curvature :} \qquad \begin{array}{l} p < 3 : U \ll g_{\rm YM}^{2/(3-p)} N^{1/(3-p)} \\ p > 3 : U \gg g_{\rm YM}^{-2/(p-3)} N^{-1/(p-3)} \end{array} \tag{31}$$

The condition of small dilaton gives

$$e^{\phi} \sim \frac{g_{\rm YM}^{(7-p)/2} N^{(3-p)/4}}{U^{(7-p)(3-p)/4}} \ll 1 \tag{32}$$

When $p < 3$, the dilaton decays at large U. This means that the U cannot be too small. When U gets too small, we need some strongly coupled string theory. Contrarily, when $p > 3$, the dilaton grows at large U, thus we would need U not to be too big.

$$\text{Small dilaton :} \qquad \begin{array}{l} p < 3 : U \gg g_{\rm YM}^{2/(3-p)} N^{1/(7-p)} \\ p > 3 : U \ll g_{\rm YM}^{-2/(p-3)} N^{-1/(p-7)} \end{array} \tag{33}$$